

Non-crossing chords of a polygon with forbidden positions

Dongyi Wei*

Demin Zhang[†]

Dong Zhang[‡]

November 11, 2016

Abstract

In this paper we investigate non-crossing chords of simple polygons in the plane systematically. We first develop the Euler characteristic of a family of line-segments, and subsequently study the structure of the diagonals and epigonals of a polygon. A special phenomenon is that the Euler characteristic of a set of diagonals (or epigonals) characterizes the geometric property of polygons, such as convexity. In particular, a positive answer is given to an open problem proposed by Shephard. The main contributions of the present paper extend such research to non-crossing diagonals and epigonals with forbidden positions. We find that the Euler characteristic of diagonals (or epigonals) with forbidden positions determine the types of polygon in surprising ways. Incidentally, some kinds of generalized Catalan's number naturally arise.

Keywords: Euler characteristic, polygon, diagonal, Catalan's number.

AMS subject classifications: 51E12, 05B25, 51D20, 51E30.

1 Introduction

A *polygon* is a closed curve, composed of a finite sequence of straight line segments. These segments are called its edges, and the points where two edges meet are the polygon's vertices. For simplicity, we restrict ourselves to simple polygons (no self-intersecting) whose vertices are in general position (no three vertices are collinear).

Given a polygon P , a *chord* is a segment whose endpoints are non-consecutive vertices of P . A chord is called a *diagonal* (resp., *epigonal*) if it lies in the interior (resp., exterior) of P .

Let $|P|$ denote the number of vertices of P . Let d_1 be the number of diagonals, d_2 be the number of disjoint pairs of diagonals, and, in general, d_i be the number of sets of i diagonals of the polygon which are pairwise non-crossing. The number e_i are defined in a similar manner for epigonals.

Theorem 1. *Let P_n be a simple polygon with $n = |P_n|$ vertices in general position. Then we have:*

(A) P_n is a convex polygon if and only if

$$d_1 - d_2 + d_3 - \cdots + (-1)^n d_{n-3} = 1 + (-1)^n \quad (1.1)$$

(B) P_n is a non-convex polygon if and only if

$$d_1 - d_2 + d_3 - \cdots + (-1)^n d_{n-3} = 1 \quad (1.2)$$

if and only if

$$e_1 - e_2 + e_3 - \cdots + (-1)^n e_{n-3} = 1 \quad (1.3)$$

*School of Mathematical Sciences and BICMR, Peking University, Beijing 100871, P. R. China.

Email addresses: jnwdyi@163.com (Dongyi Wei).

[†]Zhaipo middle school, Xinxiang 453700, Henan, P. R. China. Email addresses: 13569444933@139.com (Demin Zhang).

[‡]LMAM and School of Mathematical Sciences, Peking University, Beijing 100871, P. R. China.

Email addresses: dongzhang@pku.edu.cn or 13699289001@163.com (Dong Zhang).

Theorem 1 (A) was first proved by Lee [4], and an alternative easier proof could be found in [6]. In this article, we generalize Theorem 1 (A) to Theorem 4. Theorem 1 (B) is an open problem proposed by Shephard [5], and the first proof of (1.2) was given by Braun and Ehrenborg [7]. In Section 2, we will prove (1.3) and reprove (1.2), and thus complete the proof of Theorem 1.

This paper devotes to non-crossing chords of simple polygons with forbidden positions. Let F be a set of finite points which are in general position in the plane, and let M be a set of some line-segments with end-points in F . The family of the sets of non-crossing segments of M is

$$NC[M] = \{J \subset M : \text{the segments in } J \text{ are pairwise non-crossing}\} \cup \{\emptyset\}$$

and the related counting numbers are $f_i(M) = \#\{J \in NC[M] : \#J = i\}$, $i = 0, 1, \dots$. Here, $f_0(M) = 1$, and $\#$ is the counting function acting on finite sets. Denote by $\chi(M) := \sum_{i=0}^{\infty} (-1)^i f_i(M)$ the Euler characteristic of M . Now we restrict ourselves to a special class of segments — diagonals and epigonals of a polygon.

Definition 1. Let P be a simple polygon whose vertices are in general position. Denote $M_d(P)$ the set of diagonals of P , and $M_e(P)$ the set of epigonals of P . We usually write M_d and M_e for short. Let $d_i = f_i(M_d)$ and $e_i = f_i(M_e)$, $i = 0, 1, \dots$. Obviously, $\chi(M_d) = \sum_{i=0}^{\infty} (-1)^i d_i$ and $\chi(M_e) = \sum_{i=0}^{\infty} (-1)^i e_i$.

We observe that every $J \in NC[M_d]$ provides a partition of P by non-crossing diagonals. Given $M \subset M_d$, let

$$NC_c[M] = \{J \in NC[M] : J \text{ provides a convex partition of } P\}$$

and let $NC_{nc}[M] = NC[M] \setminus NC_c[M]$. It is noteworthy that the Euler characteristic of a set of diagonals only involves the corresponding convex partitions (see Theorem 2). This plays a central role in the development of our ideas and results.

Theorem 2. Let $J \in NC[M_d]$. Then

$$\chi(M_d \setminus J) = (-1)^{|P|+1} \sum_{I \in NC_c[J]} (-1)^{\#I}.$$

(1) If $J \in NC_{nc}[M_d]$, then $\chi(M_d \setminus J) = 0$.

(2) Let $J \in NC_c[M_d]$. Then the following statements hold.

(a) If J has a unique minimal subset $J' \in NC_c[J]$, then $\chi(M_d \setminus J) = \begin{cases} 0, & \text{if } J' \neq J \\ (-1)^{|P|+1+\#J}, & \text{if } J' = J. \end{cases}$

(b) If J is not the union of all the minimal sets in $NC_c[J]$, then $\chi(M_d \setminus J) = 0$.

(c) If $J' \subset \bigcap_{I \in NC_c[J]} I$, then $\chi(M_d \setminus J) = \prod_{k=1}^m \chi(M_d(P^k) \setminus J)$, where $m = \#J' + 1$, and P^1, \dots, P^m are the sub-polygons divided by J' .

(3) For any $l \in \mathbb{Z}$, there exists a polygon P and $J \in NC[M_d]$ such that $\chi(M_d \setminus J) = l$.

Theorem 2 could be used to determine the type of polygons with $\chi(M_d \setminus J) \neq 0$ for some fixed $J \in NC[M_d]$. Thanks to Theorem 2 (3), the values Euler characteristic of a set of diagonals can take are abundant. We provide Theorem 3 as a non-trivial application which also possesses independent interest in the study of typical polygon shapes.

Theorem 3. Linearly order the vertices of a polygon P in counter-clockwise direction, A_1, A_2, \dots, A_n , where $n := |P| \geq 5$. Given $i \in \{1, 2, \dots, n\}$, we have the following statements.

(A) $\chi(M_d \setminus \{A_i A_j : j \neq i-1, i, i+1\}) \neq 0 \Leftrightarrow P$ belongs to Class 1 or Class 2 or Class 6 (see Figs. 1, 2 and 6).

(B) $\chi(M_e \setminus \{A_i A_j : j \neq i-1, i, i+1\}) \neq 0 \Leftrightarrow P$ is convex or belongs to Class 3 (see Fig. 3).

(C) $\chi(M_d \setminus \{A_{i-1} A_{i+1}\}) \neq 0 \Leftrightarrow P$ belongs to Class 4 (see Fig. 4).

(D) $\chi(M_e \setminus \{A_{i-1} A_{i+1}\}) \neq 0 \Leftrightarrow P$ is convex or belongs to Class 2 or Class 5 (see Figs. 2 and 5).

Polygon Class 1. This is a special family of non-convex polygons with only one angle large than π . For detailed descriptions, these polygons possess the properties that $\angle A_{i+1}A_iA_{i-1} > \pi > \angle A_{i+2}A_iA_{i-2}$, and $(\angle A_{i+2}A_iA_{i-1} - \pi)(\angle A_{i+1}A_iA_{i-2} - \pi) > 0$ (see Figs. 1(a) and 1(b)).

The class of such polygons is a special case of Class 5 with a restriction that the unique special vertex A_i lies in the region I or region III (see Fig. 1(c)).

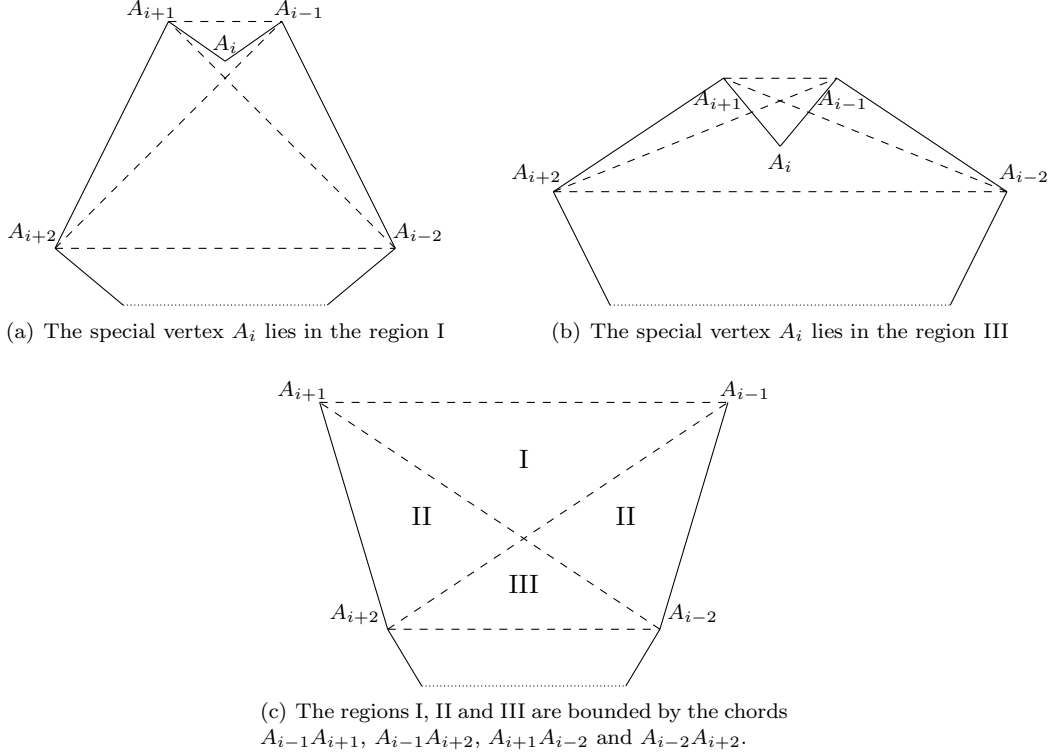


Figure 1: Figures used in Polygon Class 1.

Polygon Class 2. This is the family of all non-convex polygons with only three (consecutive) angles which are less than π . Such polygon region can be obtained by deleting a convex polygon region from a triangle region (see Fig. 2).

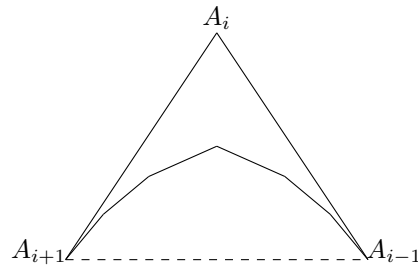


Figure 2: A figure used in Polygon Class 2. In this polygon, only the angles at the three vertices A_{i+1} , A_i and A_{i-1} are less than π .

Polygon Class 3. This family of polygon regions can be obtained by deleting a triangle region or a polygon region in Class 2 along an edge (or two neighbouring edges) of a convex polygon region (see Fig. 3).

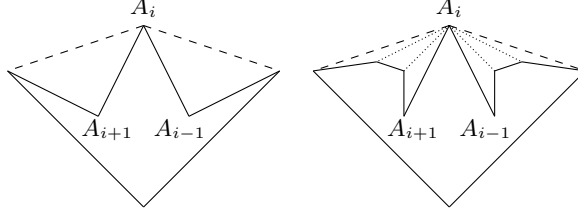


Figure 3: Figures used in Polygon Class 3. Such polygons satisfy $\angle A_{i-1}A_iA_{i+1} < \pi$.

Polygon Class 4. This family of polygon regions can be obtained by gluing a triangle region and a convex polygon region along the edge $A_{i+1}A_{i-1}$ (see Fig. 4).

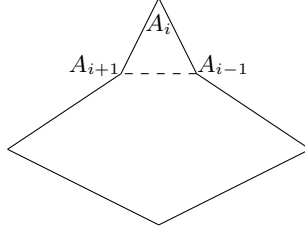


Figure 4: A figure used in Polygon Class 4. Such polygons satisfy $\angle A_{i-1}A_iA_{i+1} < \pi$.

Polygon Class 5. This family of polygon regions can be obtained by deleting a triangle region from a convex polygon region along the edge $A_{i+1}A_{i-1}$ (see Fig. 5).

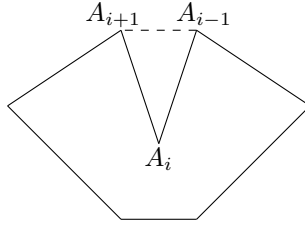


Figure 5: A figure used in Polygon Class 5.

Polygon Class 6. This family of polygon regions can be obtained by gluing a polygon region in Class 2 and a polygon region in Class 1 along the edge A_iA_{i-1} or A_iA_{i+1} (see Fig. 6).

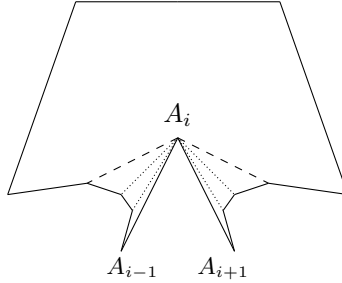


Figure 6: A figure used in Polygon Class 6.

Finally, we concentrate on some polygons with restricted number of vertices, which can be viewed as a generalization of Theorem 1 (A).

Definition 2. Given $a \in \mathbb{N}^+$ and a polygon P with $|P| = a(n+1) + 2$ for some $n \in \mathbb{N}$, a diagonal of P is said to be an a -diagonal if there are ka vertices between the diagonal's endpoints for some $k \in \mathbb{N}^+$. Let $M_d^a = M_d^a(P)$ collect the set of a -diagonals of P .

Theorem 4. Given $a, n \in \mathbb{N}^+$, let P be a convex polygon with $a(n+1) + 2$ vertices, and let $d_i(n, a) = f_i(M_d^a)$, $i = 1, 2, \dots$. Then we have $\chi(M_d^a) = (-1)^n d_n(n, a-1)$,

$$d_k(n, a) = \frac{a(n+1) + 2}{2k} \sum_{i_1+i_2=n-1} \sum_{j_1+j_2=k-1} d_{j_1}(i_1, a) d_{j_2}(i_2, a),$$

and $d_k(n, a) = \frac{1}{k+1} \binom{a(n+1)+k+1}{k} \binom{n}{k}$ for any $k \in \mathbb{N}^+$.

2 Euler characteristic for segments and the proof of Theorem 1 (B)

First we give some basic and elementary facts which are interesting and useful in the sequel.

Remark 1. Some remarks on Theorem 1 are presented below.

1. $d_0 = e_0 = 1$.
2. d_1 (resp. e_1) is the number of diagonals (resp. epigonals) of P_n .
3. $e_1 > 0 \Leftrightarrow P_n$ is non-convex.
4. d_{n-3} is the number of triangulations of P_n .
5. If $i > n$, then $d_i = e_i = 0$.
6. For each $n \geq 0$, there exists a polygon P_n such that $d_{n-3} = 1$. Polygons in Class 3 are some of these polygons.

Proposition 1. If $n \geq 4$, then P_n has diagonals, i.e., $d_1 \geq 1$.

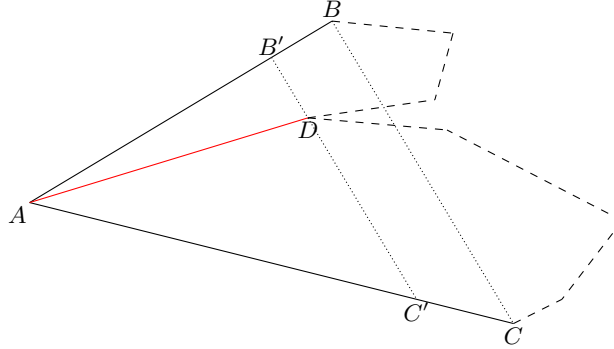


Figure 7: A figure used in Proposition 1.

Proof. We set $A_0 = A_n$ and $A_{n+1} = A_1$. Since $\sum_{i=1}^n \angle A_{i-1} A_i A_{i+1} = (n-2)\pi$, there exists $i_0 \in \{1, 2, \dots, n\}$ such that $\angle A_{i_0-1} A_{i_0} A_{i_0+1} \leq \frac{(n-2)\pi}{n} < \pi$. For simplicity, we let $B = A_{i_0-1}$, $A = A_{i_0}$ and $C = A_{i_0+1}$. Then $\angle BAC < \pi$, and the segment BC is not an edge of P_n (otherwise, $P_n = \triangle ABC$ and this contradicts with $n \geq 4$).

If BC is a diagonal, then there is nothing need to show.

If BC is not a diagonal, then there is a vertex D inside $\triangle ABC$ with greatest distance to BC (see Fig. 7). It is easy to check that AD is a diagonal. AD lies in the polygon region, which means that AD is a diagonal. \square

Proposition 2. For any set of non-crossing diagonals $J \in NC[M_d]$, there exists $J' \supset J$ which divides P_n into triangles. Particularly, for any $n \geq 3$, $d_{n-3} \geq 1$.

Proof. Note that J divides P into some polygons, which can be denoted by P_1, \dots, P_k . Clearly, each diagonal of a sub-polygon P_i is a diagonal of P , $i \in \{1, \dots, k\}$. Proposition 1 yields that if there exists P_i with $|P_i| \geq 4$, then P_i has a diagonal, and we can add the diagonal to J . Repeat the process until every sub-polygon is a triangle. At this time, we obtain J' , which provides a triangulation of P_n . Obviously, $\#J' = n - 3$ and $J' \supset J$. \square

Remark 2. If P_{n+3} is convex, then d_n is known as the Catalan number. It is well-known that $d_n = \frac{1}{n+1} \binom{2n+2}{n}$.

2.1 Euler characteristic on a set of segments in the plane

Let M be a set of segments in the plane. For $A \subset M$, let

$$NC[A] = \{J \subset A : \text{the segments in } J \text{ are pairwise non-crossing}\} \cup \{\emptyset\}$$

and let $f_i(A) = \#\{J \in NC[A] : \#J = i\}$, $i = 0, 1, \dots$. Here we set $f_0(A) = 1$. Denote by $\chi(A) := \sum_{i=0}^{\infty} (-1)^i f_i(A)$ the Euler characteristic of A .

Remark 3. (1) $\chi(\emptyset) = 1$, $\chi(\{v\}) = 0$ for $v \in A$.

(2) If $f_i(A) = 0$, then $f_{i+1}(A) = 0$.

(3) If $\#A = n$ and $i > n$, then $f_i(A) = 0$. So $\chi(A) = \sum_{i=0}^n (-1)^i f_i(A)$ is a finite sum and thus it is well-defined.

Proposition 3. If $v \in A$, then $\chi(A) = \chi(A \setminus \{v\}) - \chi(A_v)$, where A_v collects the segments in $A \setminus \{v\}$ which are non-crossing with v .

Proof. For $i \geq 1$,

$$\begin{aligned} f_i(A) &= \sum_{\#B=i, B \in NC[A]} 1 \\ &= \sum_{\#B=i, v \in B \in NC[A]} 1 + \sum_{\#B=i, v \notin B \in NC[A]} 1 \\ &= \sum_{\#B'=i-1, B' \in NC[A_v]} 1 + \sum_{\#B=i, B \in NC[A \setminus \{v\}]} 1 \\ &= f_{i-1}(A_v) + f_i(A \setminus \{v\}). \end{aligned}$$

Then we complete the proof by taking alternating sum. \square

Definition 3. Let $H \in NC[A]$. We call H a heart of A , if for any $J \in NC[A]$, there exists $J' \in NC[A]$ with $J' \supset J$ such that $J' \cap H \neq \emptyset$. If A has a heart, then we call it a star set.

Proposition 4. If A is a star set, then $\chi(A) = 0$.

Proof. We do induction on $\#A$. If $\#A = 1$, then $\chi(A) = f_0(A) - f_1(A) = 1 - 1 = 0$. Suppose that for any star set A with $\#A < n$, $\chi(A) = 0$, then for any star set A with $\#A = n$, we shall prove that $\chi(A)$ still equals to 0.

Let H be a heart of A . Thus, $H \neq \emptyset$. If $A = H$, then $\chi(A) = \sum_{i \geq 0} (-1)^i \binom{\#A}{i} = (-1 + 1)^{\#H} = 0$. Otherwise, let $v \in A \setminus H$, the Proposition 3 implies that $\chi(A) = \chi(A \setminus \{v\}) - \chi(A_v)$. Obviously, $\#A_v \leq \#(A \setminus \{v\}) = \#A - 1 = n - 1$.

For any $J \in NC[A \setminus \{v\}]$, we have $J \in NC[A]$ and thus there exists $J' \in NC[A]$ with $J' \supset J$ such that $J' \cap H \neq \emptyset$. Hence, $J' \setminus \{v\} \supset J$ and $(J' \setminus \{v\}) \cap H = J' \cap (H \setminus \{v\}) = J' \cap H \neq \emptyset$. Therefore, H is a heart of $A \setminus \{v\}$ which means that $A \setminus \{v\}$ is a star set.

For any $J \in NC[A_v]$, we have $J \cup \{v\} \in NC[A]$ and thus there exists $u \in H$ such that $J \cup \{v\} \cup \{u\} \in NC[A]$ and thus $u \in A_v$. Let $J' = J \cup \{v\} \cup \{u\}$. Then $J' \cap A_v \in NC[A_v]$ and $J' \cap A_v \supset J$, and $\emptyset \neq (J' \cap A_v) \cap H \cap A_v \ni u$. Therefore, $H \cap A_v$ is a heart of A_v , and hence A_v is a star set.

By the hypothesis of induction, we have $\chi(A \setminus \{v\}) = 0$ and $\chi(A_v) = 0$. Therefore, $\chi(A) = 0$. \square

2.2 A proof of Shephard's open problem (i.e., Theorem 1 (B))

Proof of Theorem 1 (B). Since P is non-convex, it has more than three vertices. We assume $|P| \geq 4$ and $\angle A > \pi$. Let H be a set of diagonals with an end-point A_1 (see Fig. 8). Then $H \in NC[M_d]$.

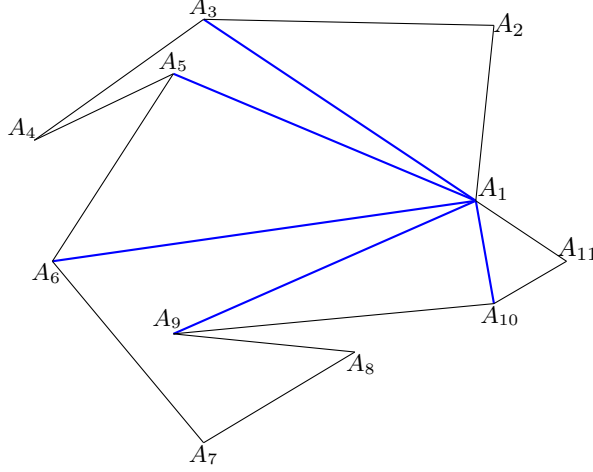


Figure 8: A figure used in the proof of Theorem 1 (B). In this polygon, we can take $H = \{A_1A_3, A_1A_5, A_1A_6, A_1A_9, A_1A_{10}\}$.

For any $J \in NC[M_d]$, by Proposition 2, there exists $J' \in NC[M_d]$ such that $J' \supset J$ and J' divides P into triangles. Since the angle A can not be an angle of a triangle, there is someone (a diagonal) in J' such that A is its end-point. Therefore $H \cap J' \neq \emptyset$. So M_d is a star set, and then by Proposition 4, we get $\chi(M_d) = 0$.

For the case of M_e , note that there exists an epigonal as a side of the convex hull of P . So it is non-crossing with other epigonals. This means that such epigonal is a heart of M_e . Consequently, M_e is a star set, and by Proposition 4, we get $\chi(M_e) = 0$. Combining with Theorem 1 (A), we complete the proof. \square

Finally, we show a generalization of (1.3).

Proposition 5. Consider the set F of finite points in the plane, and the set $S_2(F)$ of all the line-segments whose end-points lie in F . Let $S \subset S_2(F)$. If S contains an edge of the convex polygon $P_{\text{conv}}(F)$, then $\chi(S) = 0$. Here, $P_{\text{conv}}(F)$ is the boundary polygon of the convex hull $\text{conv}(F)$.

Let F be the collections of vertices of a non-convex polygon P , and let $S = M_e(P)$. Then Proposition 5 immediately implies $\chi(M_e) = 0$.

3 Some useful lemmas and the proof of Theorem 2

Lemma 1. Let $J \subset M_d$ be a set of pairwise non-crossing diagonals. For $I \subset J$, I divides P into $1 + \#I$ sub-polygons, denoted by $P_{I,k}$, $k = 1, 2, \dots, \#I + 1$. Then $\chi(M_d \setminus J) = \sum_{I \subset J} \prod_{k=1}^{\#I+1} \chi(M_d(P_{I,k}))$, where $M_d(P_{I,k})$ is the set of diagonals of $P_{I,k}$.

Proof. We classify the sets in $NC[M_d]$ via the intersection with J . It follows from the principle of inclusion-exclusion that

$$\begin{aligned} f_j(M_d \setminus J) &= \sum_{S \in NC[M_d], \#S=j, S \cap J = \emptyset} 1 \\ &= \sum_{S \in NC[M_d], \#S=j} 1 + \sum_{I \subset J, 1 \leq \#I \leq j} (-1)^{\#I} \sum_{S \in NC[M_d], \#S=j, S \cap J \supset I} 1 \\ &= f_j(M_d) + \sum_{I \subset J, 1 \leq \#I \leq j} (-1)^{\#I} f_{j-\#I}(\widehat{M_d \setminus I}) \end{aligned}$$

$$= \sum_{I \subset J, \#I \leq j} (-1)^{\#I} f_{j-\#I}(\widehat{M_d \setminus I}),$$

where $\widehat{M_d \setminus I}$ denotes the set of diagonals which is non-crossing with the diagonals in I . Then, according to the definition of Euler characteristic and the above equality, we have

$$\begin{aligned} \chi(M_d \setminus J) &= \sum_{j=0}^{\infty} (-1)^j f_j(M_d \setminus J) \\ &= \sum_{j=0}^{\infty} (-1)^j \sum_{I \subset J, \#I \leq j} (-1)^{\#I} f_{j-\#I}(\widehat{M_d \setminus I}) \\ &= \sum_{I \subset J} \sum_{j=\#I}^{\infty} (-1)^{j-\#I} f_{j-\#I}(\widehat{M_d \setminus I}) \\ &= \sum_{I \subset J} \chi(\widehat{M_d \setminus I}) \\ &= \sum_{I \subset J} \prod_{k=1}^{\#I+1} \chi(M_d(P_{I,k})). \end{aligned}$$

The last equality is a direct consequence of the product formula of Euler characteristic. \square

A direct calculation following Lemma 1 gives

$$\begin{aligned} \chi(M_d \setminus J) &= \sum_{I \subset J} \prod_{k=1}^{\#I+1} \chi(M_d(P_{I,k})) \\ &= \sum_{I \subset J, P_{I,k} \text{ convex}, \forall k} \prod_{k=1}^{\#I+1} (-1)^{|P_{I,k}|+1} \\ &= \sum_{I \subset J, P_{I,k} \text{ convex}, \forall k} (-1)^{\sum_{k=1}^{\#I+1} (|P_{I,k}|+1)} \\ &= \sum_{I \in NC_c[J]} (-1)^{|P|+2\#I+\#I+1} \\ &= (-1)^{|P|+1} \sum_{I \in NC_c[J]} (-1)^{\#I}. \end{aligned}$$

So, we complete the proof of the main part of Theorem 2. Next, we focus on the other parts.

Corollary 1. *If P is convex, and $J \in NC[M_d] \setminus \{\emptyset\}$, then $\chi(M_d \setminus J) = 0$.*

Proof. Note that $\sum_{I \in NC_c[J]} (-1)^{\#I} = \sum_{I \subset J} (-1)^{\#I} = (-1+1)^{\#J} = 0$. \square

The following Lemma 2 is another form of Theorem 2.

Lemma 2. *Let $J \subset M_d$ be a nonempty subset of pairwise non-crossing diagonals. Then $\chi(M_d \setminus J) = (-1)^{|P|} \sum_{I \in NC_{nc}[J]} (-1)^{\#I}$.*

Proof. Note that $\sum_{I \in NC[J]} (-1)^{\#I} = \sum_{I \subset J} (-1)^{\#I} = (-1+1)^{\#J} = 0$. Thus, by Theorem 2, we have

$$\begin{aligned} \chi(M_d \setminus J) &= (-1)^{|P|+1} \sum_{I \in NC_c[J]} (-1)^{\#I} \\ &= (-1)^{|P|+1} \left(\sum_{I \in NC[J]} (-1)^{\#I} - \sum_{I \in NC_c[J]} (-1)^{\#I} \right) \end{aligned}$$

$$= (-1)^{|P|} \sum_{I \in NC_{nc}[J]} (-1)^{\#I}.$$

□

Proposition 6. *If J divides P into sub-polygons containing non-convex one, then $\chi(M_d \setminus J) = 0$.*

Proof. The case of $J = \emptyset$ reduces to Theorem 1 (B). We suppose that $J \neq \emptyset$. Since $J \in NC_{nc}[M_d]$, it is easy to check that $NC_{nc}[J] = NC[J]$. Thus, combining with Lemma 2, we immediately obtain

$$\begin{aligned} \chi(M_d \setminus J) &= (-1)^{|P|} \sum_{I \in NC_{nc}[J]} (-1)^{\#I} \\ &= (-1)^{|P|} \sum_{I \subset J} (-1)^{\#I} \\ &= (-1)^{|P|} (-1 + 1)^{\#J} = 0. \end{aligned}$$

□

Proposition 7. *Suppose J divides P into convex polygons. Assume that there exists the unique minimal subset $J_c \subset J$ such that P can be divided by J_c into convex sub-polygons. Then $\chi(M_d \setminus J) = 0$ if and only if $J_c \neq J$. Besides, if J provides a minimal convex partition by non-crossing diagonals, then $\chi(M_d \setminus J) = (-1)^{|P| + \#J + 1}$.*

Proof. Since J_c is the unique minimal subset of J which divides P into convex polygons, for $I \subset J$, I divides P into convex polygons if and only if $J_c \subset I$. Combining with Lemma 2, we immediately obtain

$$\begin{aligned} \chi(M_d \setminus J) &= (-1)^{|P|+1} \sum_{I \in NC_c[M_d]} (-1)^{\#I} \\ &= (-1)^{|P|+1} \sum_{J_c \subset I \subset J} (-1)^{\#I} \\ &= (-1)^{|P|+1+\#J_c} \sum_{I' \subset J \setminus J_c} (-1)^{\#I'} \\ &= (-1)^{|P|+1+\#J_c} \begin{cases} (-1+1)^{\#(J \setminus J_c)}, & \text{if } J \setminus J_c \neq \emptyset \\ 1, & \text{if } J \setminus J_c = \emptyset \end{cases} \\ &= \begin{cases} 0, & \text{if } J_c \neq J \\ (-1)^{|P|+1+\#J}, & \text{if } J_c = J. \end{cases} \end{aligned}$$

□

Proposition 8. *Let $J \in NC_c[M_d]$ and $J' \subset J$ satisfy $J' \subset I, \forall I \in NC_c[J]$. Then $\chi(M_d(P) \setminus J) = \prod_{k=1}^m \chi(M_d(P_k) \setminus J_k)$, where $m = \#J' + 1$, and P_1, \dots, P_m are the sub-polygons divided by J' and $J_k = (J \setminus J') \cap M_d(P_k)$, $k = 1, \dots, m$.*

Proof. Let $NC_c[J_k, M_d(P_k)] = \{I \subset J_k : I \text{ divides } P_k \text{ into convex polygons}\}$. According to Lemma 2, we obtain

$$\begin{aligned} \chi(M_d \setminus J) &= (-1)^{|P|+1} \sum_{I \in NC_c[J]} (-1)^{\#I} \\ &= (-1)^{|P|+1} \sum_{I_k \in NC_c[J_k, M_d(P_k)], k=1, \dots, m} (-1)^{\#J' + \sum_{k=1}^m \#I_k} \\ &= (-1)^{|P|+1+\#J'} \prod_{k=1}^m \sum_{I_k \in NC_c[J_k, M_d(P_k)]} (-1)^{\#I_k} \\ &= (-1)^{|P|+1+\#J' - \sum_{k=1}^m (|P_k|+1)} \prod_{k=1}^m (-1)^{|P_k|+1} \sum_{I_k \in NC_c[J_k, M_d(P_k)]} (-1)^{\#I_k} \end{aligned}$$

$$= \prod_{k=1}^m \chi(M_d(P_k) \setminus J_k).$$

□

In Proposition 8, we can take $J' = \bigcap_{I \in NC_c[J]} I$.

Lemma 3. Assume that P and its convex hull exactly bound m polygons, denoted by P^1, \dots, P^m . Let $J \subset M_e$ be a subset of pairwise non-crossing epigonals. Then $\chi(M_e \setminus J) = \prod_{k=1}^m \chi(M_d(P^k) \setminus J)$.

Proof. We omit the proof. □

Definition 4. Given a non-convex polygon P , $J \in NC_c[M_d]$ and $I \subset J$, let

$$\xi(I) = \begin{cases} 0, & \text{if } I \in NC_{nc}[J] \setminus \{\emptyset\} \text{ or } I \in NC_c[J] \setminus \{J\}, \\ 1, & \text{if } I = \emptyset \text{ or } I = J. \end{cases}$$

Proposition 9. Let P be a non-convex polygon and $J \in NC_c[M_d]$. Suppose $J_1, \dots, J_m \in NC_c[J]$ are all the minimal sets. Then

$$\chi(M_d \setminus J) = (-1)^{|P|+\#J} \sum_{k=1}^m (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq m} \xi(J_{i_1} \cup \dots \cup J_{i_k}).$$

Assume $I_1, \dots, I_m \in NC_{nc}[J]$ are all the maximal sets. Then

$$\chi(M_d \setminus J) = (-1)^{|P|} \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \xi(I_{i_1} \cap \dots \cap I_{i_k}).$$

Proof. Given $I \in NC_c[J]$, let $\eta(I) = \sum_{I \subset I' \subset J} (-1)^{\#I'}$. Then $\eta(I) = \begin{cases} 0, & I \neq J \\ (-1)^{\#J}, & I = J \end{cases} = (-1)^{\#J} \xi(I)$.

Let $J_1, \dots, J_m \in NC_c[J]$ be all the minimal sets, i.e., for any $I \in NC_c[J]$, there exists $i \in \{1, \dots, m\}$ such that $I \supset J_i$. It follows from Theorem 2 and the principle of inclusion-exclusion that

$$\begin{aligned} \chi(M_d \setminus J) &= (-1)^{|P|+1} \sum_{I \in NC_c[J]} (-1)^{\#I} \\ &= (-1)^{|P|+1} \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \eta(J_{i_1} \cup \dots \cup J_{i_k}) \\ &= (-1)^{|P|+\#J} \sum_{k=1}^m (-1)^k \sum_{1 \leq i_1 < \dots < i_k \leq m} \xi(J_{i_1} \cup \dots \cup J_{i_k}). \end{aligned}$$

Given $I \in NC_{nc}[J]$, then we have $\sum_{I' \subset I} (-1)^{\#I'} = \begin{cases} 0, & I \neq \emptyset \\ 1, & I = \emptyset \end{cases} = \xi(I)$.

Let $I_1, \dots, I_m \in NC_{nc}[J]$ be all the maximal sets, i.e., for any $I \in NC_{nc}[J]$, there exists $i \in \{1, \dots, m\}$ such that $I \subset I_i$. Then Lemma 2 together with the principle of inclusion-exclusion deduce that

$$\begin{aligned} \chi(M_d \setminus J) &= (-1)^{|P|} \sum_{I \in NC_{nc}[J]} (-1)^{\#I} \\ &= (-1)^{|P|} \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \xi(I_{i_1} \cap \dots \cap I_{i_k}). \end{aligned}$$

□

Remark 4. In Proposition 9, the family of the sets J_1, \dots, J_m (resp., I_1, \dots, I_m) forms a Sperner family, i.e., none of the sets is contained in another.

Corollary 2. *Let P be a non-convex polygon and $J \in NC_c[M_d]$. Let $J_1, \dots, J_m \in NC_c[J]$ be all the minimal sets. If $J_1 \cup \dots \cup J_m \neq J$, then $\chi(M_d \setminus J) = 0$.*

Proof. Since $J_1 \cup \dots \cup J_m \neq J$, for any $1 \leq i_1 < \dots < i_k \leq m$, $J_{i_1} \cup \dots \cup J_{i_k} \neq J$. Thus by Definition 4, $\xi(J_{i_1} \cup \dots \cup J_{i_k}) = 0$, and Proposition 9 then implies $\chi(M_d \setminus J) = 0$. \square

The following result is an analogue of Corollary 2.

Corollary 3. *Let P be a non-convex polygon and $J \in NC_c[M_d]$. Let $I_1, \dots, I_m \in NC_{nc}[J]$ be all the maximal sets. If $I_1 \cap \dots \cap I_m \neq \emptyset$, then $\chi(M_d \setminus J) = 0$.*

Now we consider the statement 3 in Theorem 2. The statements 1 and 2 of Theorem 2 give the examples of the case $l \in \{-1, 0, 1\}$ in the statement 3 (for details, in the statement 1 we can take P non-convex and $J = \emptyset$, and in the statement 2 we can take P convex and $J = \emptyset$). Therefore we only need to consider the case $|l| > 1$. We first consider $l > 1$.

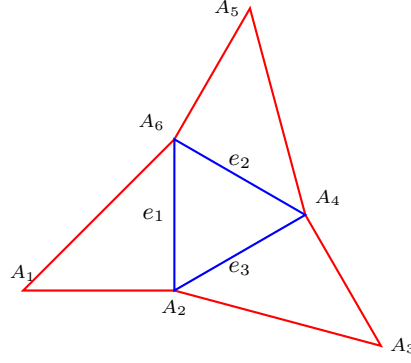


Figure 9: An example of $\chi(M_d(P) \setminus J) = 2$ used in the proof of the statement 3 of Theorem 2. Here P is the polygon with 6 red edges and J is the set of 3 blue non-crossing diagonals.

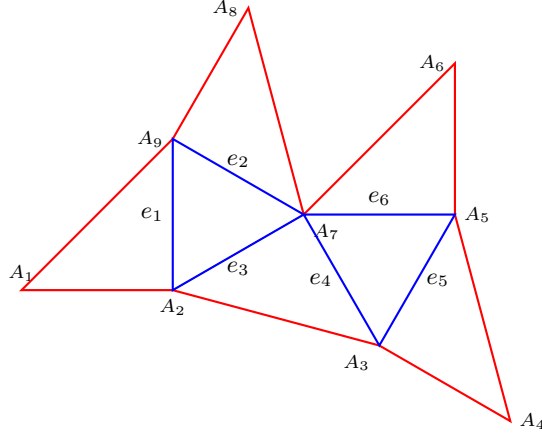


Figure 10: An example of $\chi(M_d(P) \setminus J) = 3$ used in the proof of the statement 3 of Theorem 2. Here P is the polygon with 9 red edges and J is the set of 6 blue non-crossing diagonals.

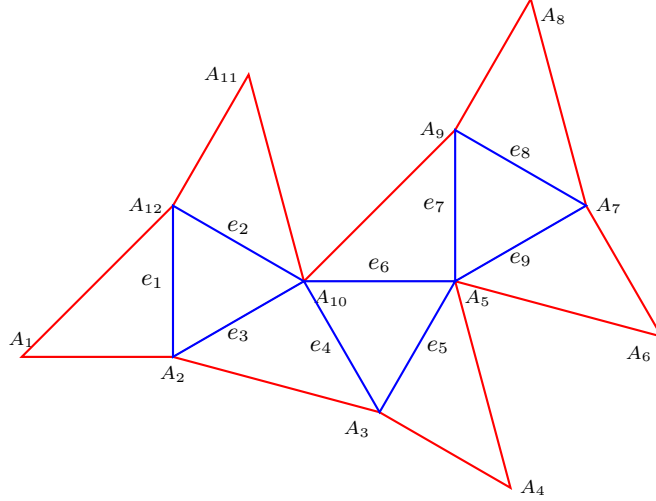


Figure 11: An example of $\chi(M_d(P) \setminus J) = 4$ used in the proof of the statement 3 of Theorem 2. Here P is the polygon with 12 red edges and J is the set of 9 blue non-crossing diagonals.

Construct P with $|P| = 3|l|$ and linearly order the vertices of P in counter-clockwise direction, $A_1, A_2, \dots, A_{3|l|}$ (see Figs. 9,10,11,12 for $l = 2, 3, 4, 5, 6, 7, 8$, respectively). We refer readers to Appendix for the detailed information of such polygons. Set

$$X = \{\{3k+2, 3|l|-3k, 3|l|-3k-2\} : 0 \leq k \leq |l|/2-1, k \in \mathbb{Z}\} \cup \{\{3k+2, 3k, 3|l|-3k+1\} : 1 \leq k < |l|/2, k \in \mathbb{Z}\}$$

and $J = \{A_i A_j : \{i, j, k\} \in X\}$. Now we label the diagonals as

$$\begin{aligned} e_{6k+1} &= A_{3k+2} A_{3|l|-3k}, & e_{6k+2} &= A_{3|l|-3k-2} A_{3|l|-3k}, & e_{6k+3} &= A_{3k+2} A_{3|l|-3k-2}, & \forall 0 \leq k \leq |l|/2-1, k \in \mathbb{Z}; \\ e_{6k-2} &= A_{3k} A_{3|l|-3k+1}, & e_{6k-1} &= A_{3k+2} A_{3k}, & e_{6k} &= A_{3k+2} A_{3|l|-3k+1}, & \forall 1 \leq k < |l|/2, k \in \mathbb{Z}. \end{aligned}$$

Then $J = \{e_1, \dots, e_{3(|l|-1)}\}$, and we can check that for $I \subset J$, $I \in NC_c[J]$ if and only if

$$I \cap \{e_k, e_{k+1}\} \neq \emptyset, \forall 1 \leq k \leq 3|l|-4 \text{ and } I \cap \{e_{3k-2}, e_{3k}\} \neq \emptyset, \forall 1 \leq k \leq |l|-1.$$

Set $J_k = \{e_{k+1}, \dots, e_{3(|l|-1)}\}$, $0 \leq k < 3(|l|-1)$, $J_{3(|l|-1)} = \emptyset$. Given $0 < k \leq 3(|l|-1)$, for any I with $J_{k-1} \subset I \subset J$, one can verify that

$$\begin{aligned} I \setminus \{e_k\} \in NC_c[J] &\Leftrightarrow e_{k-1} \in I \in NC_c[J] &\Leftrightarrow J_{k-2} \subset I \in NC_c[J], & \text{ if } 3 \nmid k; \\ I \setminus \{e_k\} \in NC_c[J] &\Leftrightarrow \{e_{k-1}, e_{k-2}\} \subset I \in NC_c[J] &\Leftrightarrow J_{k-3} \subset I \in NC_c[J], & \text{ if } 3 \mid k. \end{aligned}$$

Set

$$a_k = \sum_{J_k \subset I \in NC_c[J]} (-1)^{\#I}, \quad 0 \leq k \leq 3(|l|-1).$$

Then $a_0 = 1$, $a_1 = 0$ and for $2 \leq k \leq 3(|l|-1)$, we have

$$\begin{aligned} a_k &= a_{k-1} + \sum_{J_{k-1} \subset I, I \setminus \{e_k\} \in NC_c[J]} (-1)^{\#(I \setminus \{e_k\})} \\ &= a_{k-1} + \begin{cases} \sum_{J_{k-2} \subset I \in NC_c[J]} (-1)^{\#(I \setminus \{e_k\})} \\ \sum_{J_{k-3} \subset I \in NC_c[J]} (-1)^{\#(I \setminus \{e_k\})} \end{cases} \\ &= \begin{cases} a_{k-1} - a_{k-2}, & 3 \nmid k, \\ a_{k-1} - a_{k-3}, & 3 \mid k. \end{cases} \end{aligned}$$

Using this formula by induction we have

$$a_{3k} = (-1)^k(k+1), \forall 0 \leq k \leq |l|, \quad a_{3k+1} = (-1)^k k, \quad a_{3k+2} = (-1)^{k+1}, \forall 0 \leq k < |l|.$$

Therefore

$$\chi(M_d(P) \setminus J) = (-1)^{|P|+1} \sum_{I \in NC_c[J]} (-1)^{\#I} = (-1)^{3|l|+1} a_{3(|l|-1)} = (-1)^{3|l|+1} (-1)^{|l|-1} |l| = |l|.$$

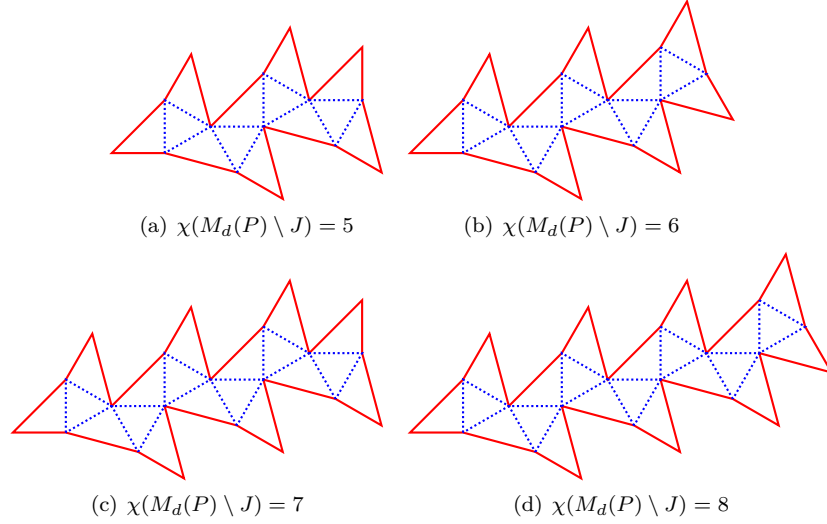


Figure 12: Examples of $\chi(M_d(P) \setminus J) \in \{5, 6, 7, 8\}$ used in the proof of the statement 3 of Theorem 2. Here P is the polygon with red edges and J is the set of corresponding blue non-crossing diagonals.

For the case of $l < -1$, we consider the polygon P' with linearly ordered vertices in counter-clockwise direction, $A'_1, A_2, \dots, A_{3|l|}, A'_{3|l|+1}$, such that A'_1 is in the segment $A_1 A_2$, $A'_{3|l|+1}$ is in the segment $A_1 A_{3|l|}$ (see Fig. 13). Then $J \in NC[M_d(P')]$ and

$$\chi(M_d(P') \setminus J) = (-1)^{|P'|+1} \sum_{I \in NC_c[J]} (-1)^{\#I} = (-1)^{3|l|+1+1} a_{3(|l|-1)} = (-1)^{3|l|} (-1)^{|l|-1} |l| = -|l| = l.$$

This completes the proof of the statement 3 in Theorem 2.

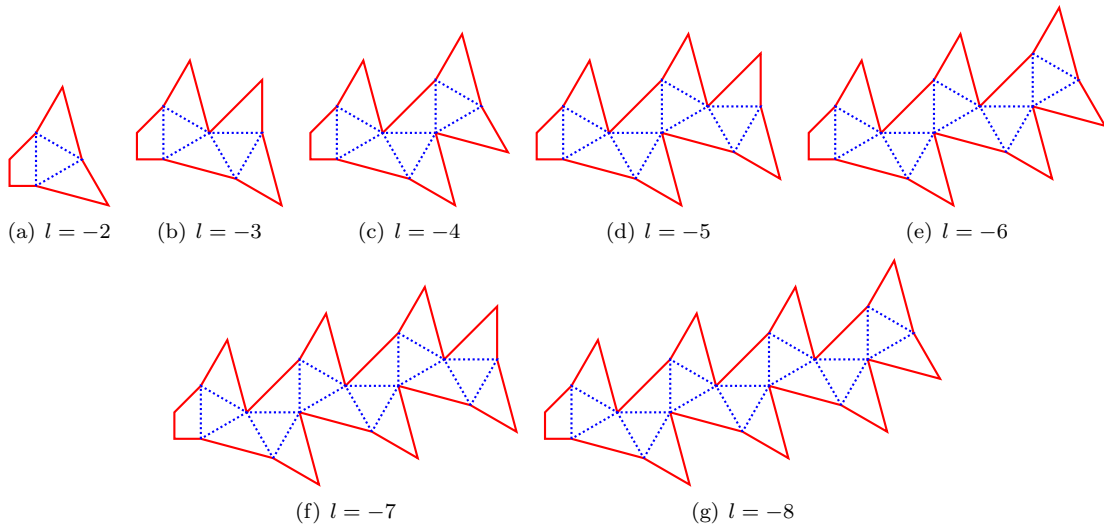


Figure 13: Examples of $\chi(M_d(P) \setminus J) = l \in \{-2, -3, \dots, -8\}$ used in the proof of the statement 3 of Theorem 2. Here P is the polygon with red edges and J is the set of corresponding blue non-crossing diagonals.

4 Proof of Theorem 3

Let $J = \{A_i A_j : j \neq i - 1, i, i + 1\}$, and without loss of generality we let $i = 1$ for simplicity.

(A) We suppose $\chi(M_d \setminus J) \neq 0$.

Claim 1 $J \subset M_d$, i.e., every chord $A_1 A_j$ is a diagonal of P , where $j \neq 1, 2, n$.

Since $\chi(M_d \setminus J) \neq 0$, Proposition 6 implies that $J \cap M_d$ divides P into convex polygons, with a common vertex A_1 . Suppose that there exists $j_1 \neq 1$ such that $A_1 A_{j_1} \notin M_d$. Then $A_1 A_{j_1}$ is not an edge of these convex sub-polygons. Note that A_{j_1} must be a vertex of a convex sub-polygon. So, $A_1 A_{j_1}$ is a diagonal of such convex sub-polygon and thus $A_1 A_{j_1}$ is a diagonal of P , which is a contradiction.

By Claim 1, we have $\angle A_3 A_2 A_1 < \pi$ and $\angle A_1 A_n A_{n-1} < \pi$ (see Fig. 14).

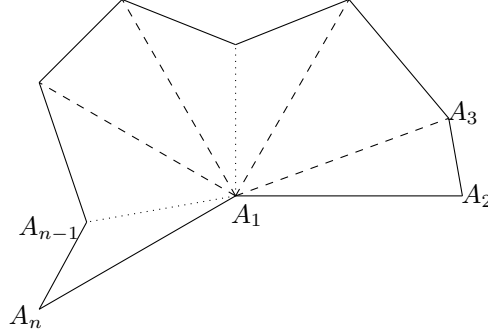


Figure 14: A figure used in the proof of Theorem 1 (B). In this polygon, all the dotted lines are collected in J , and all the red dashed lines are collected in J_c .

Let

$$J_c = \begin{cases} \{A_1 A_j : \angle A_{j-1} A_j A_{j+1} > \pi, j \neq 1, 2, n\}, & \text{if } \exists j \neq 1, 2, n \text{ s.t. } \angle A_{j-1} A_j A_{j+1} > \pi, \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is easy to see that $J_c \subset J$ (see Fig. 14).

Claim 2 If $I \subset J$ divides P into convex polygons, then $I \supset J_c$.

Suppose the contrary, that $I \not\supset J_c$, i.e., there exists $A_1 A_j \in J_c \setminus I$. Then there is a sub-polygon containing the vertices A_{j-1}, A_j, A_{j+1} . Since $\angle A_{j-1} A_j A_{j+1} > \pi$, such sub-polygon must be non-convex and this leads to a contradiction.

Claim 3 If J_c divides P into convex polygons, then $\chi(M_d \setminus J) \neq 0$ if and only if $J_c = J$.

Since J_c divides P into convex polygons, Claim 2 then implies that J_c is the minimal set of J that divides P into convex polygons. So, Proposition 7 (i.e., Theorem 2 (2)(a)) deduces Claim 3.

Now we divide the proof of Theorem 3 (A) into several cases.

Case 1. J_c divides P into convex polygons. This is equivalent to $\angle A_2 A_1 A_n < \pi$ and $\angle A_{j+1} A_j A_{j-1} > \pi$ for any $j \neq 1, 2, n$.

Since $\chi(M_d \setminus J) \neq 0$, Claim 3 implies that $J_c = J$. That is, $\angle A_{j+1} A_j A_{j-1} > \pi$ for any $j \neq 1, 2, n$. Thus, $(n-3)\pi + \angle A_3 A_2 A_1 + \angle A_2 A_1 A_n + \angle A_1 A_n A_{n-1} < \sum_{j=1}^n \angle A_{j+1} A_j A_{j-1} = (n-2)\pi$. We immediately get $\angle A_2 A_1 A_n < \pi$. So, P belongs to Class 2.

Case 2. J_c doesn't divide P into convex polygons.

In this case, $J_c \neq J$ and $\angle A_2 A_1 A_n > \pi$.

Case 2.1. $J_c = \emptyset$, i.e., $\angle A_{j+1} A_j A_{j-1} < \pi$ for any $j \neq 1, 2, n$.

In this case, combining with the fact $J \subset M_d$, we further have $\angle A_{j+1} A_j A_{j-1} < \pi$ for any $j \neq 1$. If $\angle A_2 A_1 A_n < \pi$, then P is a convex polygon. Thus, Corollary 1 deduces that $\chi(M_d \setminus J) = 0$ unless P is a triangle. Next we assume that $\angle A_2 A_1 A_n > \pi$.

Case 2.1.1. $NC_{nc}[J] = \{\emptyset\}$, i.e., $\angle A_3 A_1 A_n < \pi$ and $\angle A_2 A_1 A_{n-1} < \pi$ (see Fig. 1(a)).

In this case, $\chi(M_d \setminus J) = (-1)^{|P|}$.

Case 2.1.2. $NC_{nc}[J] \setminus \{\emptyset\} \neq \emptyset$.

Then each maximal $I \in NC_{nc}[J]$ possesses the form $\{A_1 A_3, \dots, A_1 A_i\} \cup \{A_1 A_j, \dots, A_1 A_{n-1}\}$ with $3 \leq i$ and $j \leq n-1$, and the only non-convex sub-polygon is $A_1 A_i A_{i+1} \cdots A_j$ with $\angle A_i A_1 A_j > \pi$. Here the first part

$\{A_1A_3, \dots, A_1A_i\}$ or the second part $\{A_1A_j, \dots, A_1A_{n-1}\}$ may be empty but cannot be both empty, and if the two parts are both nonempty then $i < j$. So, each maximal $I \in NC_{nc}[J]$ contains the diagonal A_1A_3 or the diagonal A_1A_{n-1} . Let $I_1, \dots, I_m \in NC_{nc}[J]$ be all the maximal sets. If $\cap_{i=1}^m I_i \neq \emptyset$, then $\xi(I_{i_1} \cap \dots \cap I_{i_k}) = 0$ for any $\{i_1, \dots, i_k\} \subset \{1, \dots, m\}$ and thus $\chi(M_d \setminus J) = 0$. So, $\cap_{i=1}^m I_i = \emptyset$, i.e., there exists $i, j \in \{1, \dots, m\}$ such that $A_1A_3 \notin I_i$ and $A_1A_{n-1} \notin I_j$, and clearly, such i and j are unique. Without loss of generality, we may assume that $A_1A_3 \notin I_m$ and $A_1A_{n-1} \notin I_1$. Then $A_1A_3 \in I_1$ and $A_1A_{n-1} \in I_m$.

If $m > 2$, then for any $2 \leq i \leq m-1$, $\{A_1A_3, A_1A_{n-1}\} \subset I_i$. Note that $\xi(I_{i_1} \cap \dots \cap I_{i_k}) = 1 \Leftrightarrow \xi(I_{i_1} \cap \dots \cap I_{i_k}) \neq 0 \Leftrightarrow \{I_1, I_m\} \subset \{I_{i_1}, \dots, I_{i_k}\}$. It follows from Proposition 9 that

$$\begin{aligned} \chi(M_d \setminus J) &= \sum_{k=1}^m (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k \leq m} \xi(I_{i_1} \cap \dots \cap I_{i_k}) = \sum_{k=2}^m (-1)^{k-1} \sum_{1 \leq i_1 < \dots < i_k = m} 1 \\ &= \sum_{k=2}^m (-1)^{k-1} \binom{m-2}{k-2} = -(-1+1)^{m-2} = 0, \end{aligned}$$

which is a contradiction.

Thus, $m = 2$, and we can assume $I_1 = \{A_1A_3, \dots, A_1A_i\}$ and $I_2 = \{A_1A_j, \dots, A_1A_{n-1}\}$. Obviously, $\angle A_3A_1A_{n-1} < \pi$, $\angle A_3A_1A_n > \pi$ and $\angle A_2A_1A_{n-1} > \pi$ (see Fig. 1(b)). In this case, $\chi(M_d \setminus J) = (-1)^{|P|}(\xi(I_1) + \xi(I_2) - \xi(I_1 \cap I_2)) = (-1)^{|P|+1}$.

Case 2.2. $J_c \neq \emptyset$.

Assume $J_c = \{A_1A_{i_1}, \dots, A_1A_{i_k}\}$, where $3 \leq i_1 < \dots < i_k \leq n-1$. For simplicity, we set $i_0 = 2$ and $i_{k+1} = n-1$. Note that $\sum_{s=0}^k \angle A_{i_s}A_1A_{i_{s+1}} = \angle A_2A_1A_n < 2\pi$. So, there is at most one $s \in \{0, 1, \dots, k\}$ such that $\angle A_{i_s}A_1A_{i_{s+1}} > \pi$. If for any $s \in \{0, 1, \dots, k\}$, $\angle A_{i_s}A_1A_{i_{s+1}} < \pi$, then J_c divides P into convex polygons, which contradicts to the assumption of Case 2.

Hence, there exists a unique $t \in \{0, 1, \dots, k\}$ such that $\angle A_{i_t}A_1A_{i_{t+1}} > \pi$ (see Fig. 15). Now we shall prove that for any $j \in \{3, \dots, i_t\} \cup \{i_{t+1}, \dots, n-2\}$, $\angle A_{j+1}A_jA_{j-1} > \pi$ and thus $J_c = \{A_1A_j : j = 3, \dots, i_t, i_{t+1}, \dots, n-2\}$. If not, then there exists $j_0 \in \{3, \dots, i_t\} \cup \{i_{t+1}, \dots, n-2\}$ such that $\angle A_{j_0+1}A_{j_0}A_{j_0-1} < \pi$, i.e., $A_1A_{j_0} \notin J_c$. Thus, for any minimal set $I \in NC_c[J]$, $A_1A_{j_0} \notin I$. Suppose $J_1, \dots, J_m \in NC_c[J]$ are all the minimal sets. Then $\cup_{i=1}^m J_i \neq J$ and thus Corollary 2 implies that $\chi(M_d \setminus J) = 0$, which is a contradiction.

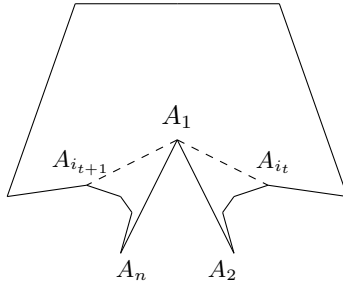


Figure 15: A figure used in the proof of Theorem 1 (B).

Then Proposition 8 implies that $\chi(M_d \setminus J) = \chi(M_d(A_1A_{i_t} \dots A_{i_{t+1}}) \setminus (J \setminus J_c))$. Note that the sub-polygon $P' := A_1A_{i_t} \dots A_{i_{t+1}}$ (see Fig. 15) fulfils the assumption of Case 2.1, i.e., $\angle A_{i_t}A_1A_{i_{t+1}} > \pi$ (fulfils the assumption of Case 2) and $\angle A_{j+1}A_jA_{j-1} < \pi$ for any $j \neq 1, i_t, i_{t+1}$ (further fulfils the assumption of Case 2.1). And note that the sub-polygons $A_1A_2 \dots A_{i_t}$ (if $i_t \neq 1, 2$) and $A_1A_{i_{t+1}} \dots A_n$ (if $i_{t+1} \neq n, 1$) satisfy the assumption of Case 1 (see Fig. 15). In consequence, P belongs to Class 6.

(B) If P_n is a convex polygon, then $M_e = \emptyset$, $\chi(M_e \setminus J) = \chi(\emptyset) = 1$, and thus the statement obviously holds.

Next we focus on the non-convex case. Then the boundary of the convex hull of P_n forms a convex polygon $P_{conv}(P_n)$ and each edge of the convex polygon is either an edge of P_n or an epigonal of P_n . Since $\chi(M_e \setminus J) \neq 0$, all the edges of the convex polygon $P_{conv}(P_n)$ which is an epigonal of P_n must belong to $J = \{A_1A_j : j \neq 1, 2, n\}$. Thus, A_1 is a vertex of $P_{conv}(P_n)$ and the number of such epigonals which are edges of $P_{conv}(P_n)$ is at most two.

Note that the case of one epigonal could be seen as a degenerate case. We may assume without loss of generality that there are exact two epigonals which are edges of $P\text{conv}(P_n)$ with the common vertex A_1 , denoted by $A_1A_{j_1}$ and $A_1A_{j_2}$. Then there are two polygons between P_n and $P\text{conv}(P_n)$, denoted them by P^1 and P^2 which respectively possesses the edges $A_1A_{j_1}$ and $A_1A_{j_2}$.

So, by Lemma 3, $\chi(M_e(P_n)) = \chi(M_d(P^1 \setminus J))\chi(M_d(P^2 \setminus J))$, and thus we have $\chi(M_d(P^1 \setminus J)) \neq 0$ and $\chi(M_d(P^2 \setminus J)) \neq 0$. Theorem 3 (A) shows that P^1 and P^2 must belong to Class 1 or Class 2 or Class 6. Note that the sum of the angle at A_1 of P^1 and the angle at A_1 of P^2 is less than $\angle A_{j_1}A_1A_{j_2} < \pi$. So, P^1 and P^2 must belong to Class 2, and thus P_n must belong to Class 3.

(C) If $A_2A_n \notin M_d$, then P_n is non-convex and $M_d \setminus \{A_2A_n\} = M_d$. Hence, $\chi(M_d \setminus \{A_2A_n\}) = \chi(M_d) = 0$. Next we assume that $A_2A_n \in M_d$. Then $\angle A_2A_1A_n < \pi$.

If P_n is a convex polygon, then Corollary 1 deduces that $\chi(M_d \setminus \{A_2A_n\}) = 0$.

If P_n is a non-convex polygon and $\chi(M_d \setminus \{A_2A_n\}) \neq 0$, then Proposition 6 implies that A_2A_n divides P_n into convex polygons. Hence, the sub-polygon $A_2A_3 \cdots A_n$ is convex, and as a consequence, P belongs to Class 4.

(D) If P_n is a convex polygon, then $\chi(M_e \setminus \{A_2A_n\}) \neq 0$. So, we only concentrate on the non-convex case.

If $\chi(M_e \setminus \{A_2A_n\}) \neq 0$, then A_2A_n is the unique edge of $P\text{conv}(P_n)$ that is not an edge of P_n . Such polygon must belong to Class 2 or Class 5.

For Class 2, $\chi(M_e \setminus \{A_2A_n\}) = \chi(M_d(A_2A_3 \cdots A_n)) = (-1)^n$. For Class 5, $\chi(M_e \setminus \{A_2A_n\}) = \chi(\emptyset) = 1$.

We have completed the proof of Theorem 3.

5 Proof of Theorem 4

First we give some basic and elementary facts for a -diagonals of a polygon P .

Remark 5. 1. Each diagonal is an 1-diagonal.

2. Each ab -diagonal is both a -diagonal and b -diagonal.

3. A polygon P is called an a -polygon if it has $a(n+1)+2$ vertices for some $n \in \mathbb{N}$. Every a -diagonal of P divides P into two a -polygons.

4. If P is convex and $|P| = a(n+1)+2$, then there exists n a -diagonals from one vertex.

Let $P = P_{a(n+1)+2}$ be a convex polygon with $a(n+1)+2$ vertices. Let $d_1(n, a)$ be the number of a -diagonals, $d_2(n, a)$ be the number of non-crossing pairs of a -diagonals, and, in general, $d_i(n, a)$ be the number of sets of i a -diagonals of the polygon which are pairwise non-crossing.

Proposition 10. $d_k(n, a) = \frac{1}{k+1} \binom{n}{k} \binom{a(n+1)+k+1}{k}$.

Proof. Corollary 6 in [6] gives that the number of different ways of cutting $P_{a(n+1)+2}$ into sub-polygons $P_{ai_1+2}, P_{ai_2+2}, \dots, P_{ai_{k+1}+2}$ by diagonals is always $\frac{1}{k+1} \binom{a(n+1)+k+1}{k}$, where (i_1, \dots, i_{k+1}) is a given ordered array of positive integers.

Note that $\sum_{j=1}^{k+1} |P_{ai_j+2}| = |P_{a(n+1)+2}| + 2k$, i.e., $a(i_1 + \dots + i_{k+1}) + 2(k+1) = a(n+1) + 2 + 2k$, and this is equivalent to $i_1 + i_2 + \dots + i_{k+1} = n+1$. Since the number of positive integer solutions of $i_1 + i_2 + \dots + i_{k+1} = n+1$ is $\binom{n}{k}$, we have $d_k(n, a) = \frac{1}{k+1} \binom{a(n+1)+k+1}{k} \binom{n}{k}$, which completes the proof. \square

Proposition 11. For any $a \in \mathbb{N}^+$, we have

$$\sum_{k=1}^n (-1)^{k-1} d_k(n, a) = 1 + (-1)^{n+1} \frac{\binom{a(n+1)}{n}}{n+1} = 1 + (-1)^{n+1} d_n(n, a-1)$$

Proposition 11 can be proved by modifying the ideas in [6].

Proof.

$$\begin{aligned}
\sum_{k=1}^n (-1)^{k-1} d_k(n, a) &= \sum_{k=1}^n (-1)^{k-1} \frac{\binom{n}{k}}{k+1} \binom{a(n+1)+k+1}{k} \\
&= \sum_{k=1}^n (-1)^{k-1} \frac{\binom{n+1}{k+1}}{n+1} \operatorname{Res}_0 \left(\frac{(1+u)^{a(n+1)+k+1}}{u^{k+1}} \right) \\
&= \frac{1}{n+1} \operatorname{Res}_0 \left((1+u)^{a(n+1)} \sum_{k=1}^n (-1)^{k-1} \binom{n+1}{k+1} \frac{(1+u)^{k+1}}{u^{k+1}} \right) \\
&= \frac{1}{n+1} \operatorname{Res}_0 \left((1+u)^{a(n+1)} \left[\left(1 - \frac{u+1}{u}\right)^{n+1} - \left(1 - (n+1) \frac{u+1}{u}\right) \right] \right) \\
&= \frac{1}{n+1} \operatorname{Res}_0 \left((1+u)^{a(n+1)} \left[\left(-\frac{1}{u}\right)^{n+1} - 1 + (n+1) \frac{u+1}{u} \right] \right) \\
&= \frac{1}{n+1} \left\{ (-1)^{n+1} \operatorname{Res}_0 \left(\frac{(1+u)^{a(n+1)n}}{u^{n+1}} \right) + (n+1) \operatorname{Res}_0 \left(\frac{(1+u)^{a(n+1)+1}}{u} \right) \right\} \\
&= \frac{1}{n+1} \left\{ (-1)^{n+1} \binom{a(n+1)}{n} + (n+1) \cdot 1 \right\} \\
&= 1 + (-1)^{n+1} \frac{\binom{a(n+1)}{n}}{n+1} = 1 + (-1)^{n+1} d_n(n, a-1),
\end{aligned}$$

where $\operatorname{Res}_0(f(u))$ means the residue of the function $f(u)$ at $u = 0$. □

Remark 6. Since $d_k(n, 1) = d_k(n+1)$ and $d_n(n, 0) = 0$, Lee's theorem (i.e., Theorem 1 (A))

$$d_1 - d_2 + d_3 - \cdots + (-1)^n d_{n-1} = 1 + (-1)^n$$

is clearly a special case of Proposition 11 for $a = 1$.

Proposition 12. Given $k, n \in \mathbb{N}^+$, we have

$$d_k(n, a) = \frac{a(n+1)+2}{2k} \sum_{i_1+i_2=n-1} \sum_{j_1+j_2=k-1} d_{j_1}(i_1, a) d_{j_2}(i_2, a)$$

Remark 7. This result is a generalization of the identity of Catalan number. The proof is standard and hence we omit it.

By Proposition 10 and Proposition 12, we have the following combinatorial identity.

Proposition 13. Given $n, i \in \mathbb{N}^+$, we have

$$\frac{\binom{a(n+1)+(i+1)}{i} \binom{n}{i}}{i+1} = \frac{a(n+1)+2}{2i} \sum_{\substack{n_1+n_2=n-1, \\ i_1+i_2=i-1}} \frac{\binom{a(n_1+1)+(i_1+1)}{i_1} \binom{a(n_2+1)+(i_2+1)}{i_2} \binom{n_1}{i_1} \binom{n_2}{i_2}}{(i_1+1)(i_2+1)} \quad (5.1)$$

Remark 8. This result can be proved by PDE method and residue theorem.

Proof of Proposition 13. Let $a_{n,i} = \frac{1}{i+1} \binom{a(n+1)+(i+1)}{i} \binom{n}{i}$ and

$$F(x, y) = \sum_{n, i \geq 0} a_{n,i} x^n y^i. \quad (5.2)$$

Then $xyF^2(x, y) = \sum_{n, i \geq 0} b_{n,i} x^n y^i$. (5.1) is equivalent to $a_{n,i} = \frac{a(n+1)+2}{2i} b_{n,i}$, $n, i > 0$, which can be written as $2ia_{n,i} = (a(n+1)+2)b_{n,i}$, $n, i \geq 0$. Note that $i = 0$ or $i > n$ implies $b_{n,i} = 0$, then it follows from $\sum_{n, i \geq 0} (a(n+1)+2)b_{n,i} x^n y^i = a(x^2 y F^2)_x + 2xy F^2$ and $\sum_{n, i \geq 0} 2ia_{n,i} x^n y^i = 2y F_y$ that (5.1) is equivalent to

$a(x^2yF^2)_x + 2xyF^2 = 2yF_y$. This can be simplified as $a(2xyF^2 + 2x^2yFF_x) + 2xyF^2 = 2yF_y$, which can be further written as

$$(1+a)xF^2 + ax^2FF_x = F_y. \quad (5.3)$$

Next we use the method of characteristics to solve (5.3).

Let $x = x(y)$ solve $\frac{dx}{dy} = -ax^2F(x, y)$. Then $F = F(x(y), y)$ solves $\frac{dF}{dy} = (1+a)F^2x$, and we have $\frac{d(xF)}{dy} = F^2x^2$.

Thus we have $xF = \frac{1}{c_1 - y}$ for some $c_1 \in \mathbb{R}$, and then $\frac{dx}{dy} = \frac{-ax}{c_1 - y}$, $x = c_2(c_1 - y)^a$ for some $c_2 \in \mathbb{R}$. Therefore, $F = \frac{1}{c_2(c_1 - y)^{a+1}}$. Taking the initial datum $y = 0$ in (5.2), we have $F(x, 0) = \sum_{n \geq 0} a_{n,0}x^n = \sum_{n \geq 0} x^n = \frac{1}{1-x}$, and thus $\frac{1}{1-c_2c_1^a} = \frac{1}{c_2c_1^{a+1}}$. So, $c_2c_1^a(c_1 + 1) = 1$, $x = \frac{(c_1 - y)^a}{c_1^a(c_1 + 1)}$, $F = \frac{c_1^a(c_1 + 1)}{(c_1 - y)^{a+1}}$.

Let $t = \frac{y}{c_1}$. Then $x = \frac{t(1-t)^a}{y+t}$, $F = \frac{y+t}{(1-t)^{a+1}y}$. Let $t = xv$, we have

$$y = v((1-xv)^a - x), \quad F = \frac{1}{(1-xv)((1-xv)^a - x)} \quad (5.4)$$

According to the implicit function theorem, v and then F must be an analytic function of (x, y) for sufficiently small $|x|, |y|$, thus (5.4) gives a solution of (5.3). Next we prove that the F satisfying (5.4) must satisfy (5.2).

Let $F = \sum_{i \geq 0} f_i(x)y^i$. Then by the residue theorem, we have

$$\begin{aligned} f_i(x) &= \text{Res}_0(Fy^{-i-1}dy) = \text{Res}_0\left(\frac{(1-xv)^a x - axv(1-xv)^{a-1}}{(1-xv)((1-xv)^a - x)^{i+2}v^{i+1}}dv\right) \\ &= \text{Res}_0\left(\frac{dv}{(1-xv)^{1+a(i+1)}v^{i+1}}\left(1 - \frac{x}{(1-xv)^a}\right)^{-i-1} - \frac{axdv}{(1-xv)^{2+a(i+1)}v^i}\left(1 - \frac{x}{(1-xv)^a}\right)^{-i-2}\right) \\ &= \text{Res}_0\left(\frac{dv}{(1-xv)^{1+a(i+1)}v^{i+1}}\sum_{n=i}^{+\infty} \frac{\binom{n}{i}x^{n-i}}{(1-xv)^{a(n-i)}} - \frac{axdv}{(1-xv)^{2+a(i+1)}v^i}\sum_{n=i+1}^{+\infty} \frac{\binom{n}{i+1}x^{n-i-1}}{(1-xv)^{a(n-i-1)}}\right) \\ &= \sum_{n=i}^{+\infty} \text{Res}_0\left(\frac{dv x^{n-i} \binom{n}{i}}{(1-xv)^{1+a(n+1)}v^{i+1}}\right) - \sum_{n=i+1}^{+\infty} \text{Res}_0\left(\frac{ax^{n-i} dv \binom{n}{i+1}}{(1-xv)^{2+an}v^i}\right) \\ &= \sum_{n=i}^{+\infty} x^n \binom{n}{i} \binom{1+a(n+1)+i-1}{i} - \sum_{n=i+1}^{+\infty} ax^{n-1} \binom{n}{i+1} \binom{2+an+i-2}{i} \\ &= \sum_{n=i}^{+\infty} x^n \left(\binom{n}{i} \binom{a(n+1)+i}{i} - a \binom{n+1}{i+1} \binom{a(n+1)+i}{i}\right) \\ &= \sum_{n=i}^{+\infty} x^n \binom{n}{i} \binom{a(n+1)+i}{i} \left(1 - a \frac{n+1}{i+1} \frac{i}{a(n+1)+1}\right) \\ &= \sum_{n=i}^{+\infty} x^n \binom{n}{i} \binom{a(n+1)+i}{i} \frac{a(n+1)+i+1}{(i+1)(a(n+1)+1)} \\ &= \sum_{n=i}^{+\infty} x^n \binom{n}{i} \binom{a(n+1)+i+1}{i} \frac{1}{i+1} = \sum_{n \geq 0} x^n a_{n,i} \end{aligned}$$

Therefore, (5.2) holds, and then (5.1) holds. \square

Acknowledgement The third-named author thanks Zipei Nie for interesting discussions.

References

- [1] A. Cayley, On the partition of a polygon, *Proc. Lond. Math. Soc. (3)* **22** (1890), 237–262.
- [2] J. H. Przytycki and A. S. Sikora, Polygon dissections and Euler, Fuss, Kirkman, and Cayley numbers, *J. Combin. Theory Ser. A* **92** (2000), 68–76.

- [3] M. S. Floater and T. Lyche, Divided differences of inverse functions and partitions of a convex polygon, *Math. Comp.* **77** (2008), 2295–2308.
- [4] C. W. Lee, The associahedron and triangulations of the n -gon, *European J. Combin.* **10** (1989), 551–560.
- [5] G. C. Shephard, A polygon problem, *Amer. Math. Monthly* **102** (1995), 505–507.
- [6] Dong Zhang, Dongyi Wei and Demin Zhang, Combinatorial Enumeration of Partitions of a Convex Polygon, *J. Integer Sequences* **18** (2015), Article 15.9.4.
- [7] Benjamin Braun, Richard Ehrenborg, The complex of non-crossing diagonals of a polygon, *J. Combin. Theory Ser. A* **117** (2010), 642–649.

Appendix: the precise construction process of the polygons used in Theorem 2

We use complex coordinate. Denote $\omega_k = (1 + (-1)^k \sqrt{3}i)/2$ and set the points B_k, C_k, D_k such that $\forall k \in \mathbb{Z}$,

$$B_{2k} - B_{2k-1} = 1, B_{2k+1} - B_{2k} = e^{\frac{\pi}{6}i}, C_k - B_k = \omega_k(B_{k+1} - B_k), D_k - B_k = \omega_k(B_{k+2} - B_k).$$

Then we can choose

$$A_{3k-1}^0 = B_{2k}, A_{3k}^0 = C_{2k+1}, A_{3k+1}^0 = D_{2k+1}, A_{3k-3}^1 = C_{2k}, A_{3k-2}^1 = D_{2k}, A_{3k-1}^1 = B_{2k+1}, \forall k \in \mathbb{Z}.$$

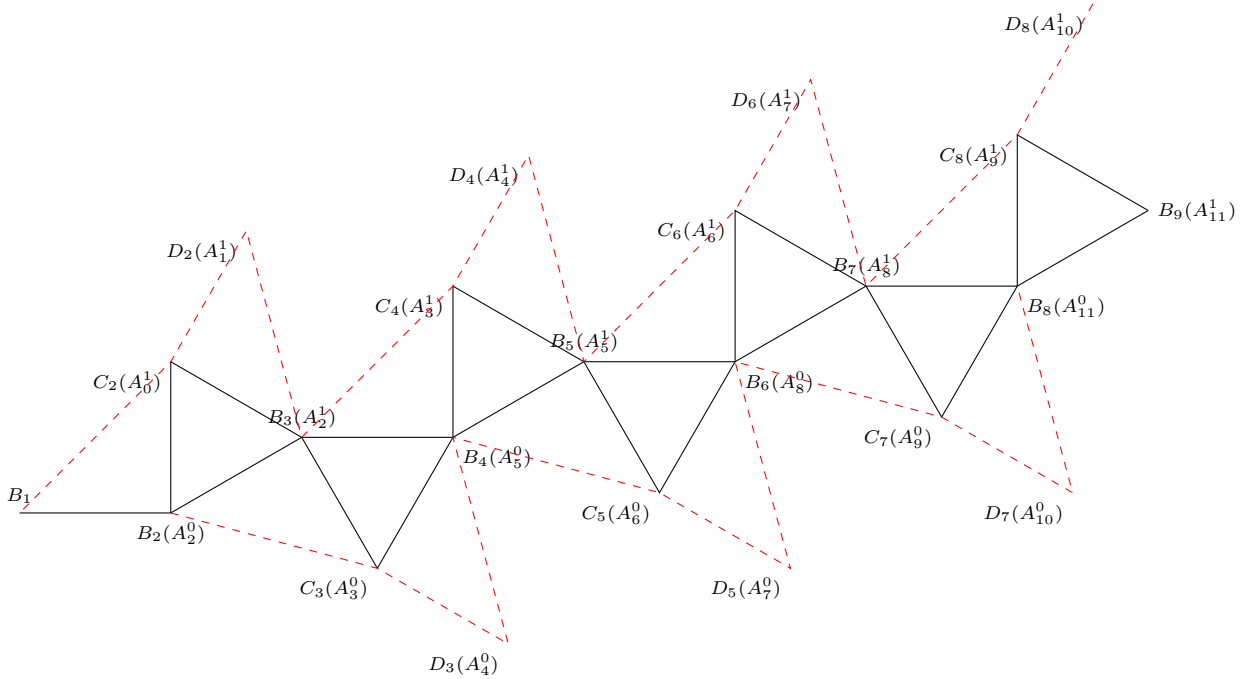


Figure 16: An figure used in Appendix.

And next we take (for the case of $l > 1$)

$$A_1^2 = B_1, A_k^2 = A_k^0, \forall 2 \leq k \leq (3l+1)/2, k \in \mathbb{Z}, A_{3l-k}^2 = A_k^1, \forall 0 \leq k \leq (3l-2)/2, k \in \mathbb{Z}.$$

Then the polygon with linearly ordered counter-clockwise vertices $A_1^2, A_2^2, \dots, A_{3l}^2$ is our desired polygon P . Although many vertices are collinear, we can avoid this by small perturbations, i.e., replace the vertices $A_1^2, A_2^2, \dots, A_{3l}^2$ by the points A_1, A_2, \dots, A_{3l} which are in general position and $|A_k - A_k^2| < 1/100, k = 1, \dots, 3l$.